Series Solutions of ODEs - Examples

(1) $y''(x) + 2x\,y'(x) - 3y(x) = 0$ (Example from class)

Assume: $y(x) = \sum_{n=0}^{\infty} c_n x^n$

$$\hookrightarrow 0 = \sum_{n=0}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} 2n c_n x^n - \sum_{n=0}^{\infty} 3c_n x^n$$

$$= \underbrace{\sum_{n=0}^{\infty} n(n-1)c_n x^{n-2}} + \sum_{n=0}^{\infty} (2n-3)c_n x^n$$

Re-index so it looks like the other sum: starts @ $n=0$ w/ powers of x^n.

$$\sum_{n=0}^{\infty} n(n-1)c_n x^{n-2} = \underset{n=0 \text{ term}}{0} + \underset{n=1 \text{ term}}{0} + \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}$$

$n \to n+2$

$$\sum_{\underbrace{n+2=2}_{n=0}}^{\infty} (n+2)(n+\cancel{2}\cancel{-1}^{1})c_{n+2}\,x^{n+\cancel{2}\cancel{-2}}$$

$$= \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2}\,x^n$$

$$0 = \sum_{n=0}^{\infty} \Big[\underbrace{(n+2)(n+1)c_{n+2} + (2n-3)c_n}\Big] x^n$$

This contains all the terms from the original 3 sums.

The equation is satisfied if

$$c_{n+2} = \frac{(3-2n)}{(n+2)(n+1)}\,c_n$$

$$\begin{cases} c_2 = \frac{3}{2}c_0 \quad c_4 = \frac{-1}{12}c_2 = -\frac{1}{8}c_0 \\ c_6 = \frac{-\cancel{5}}{6\cdot\cancel{5}}c_4 = \frac{1}{48}c_0 \quad \ldots \\ c_3 = \frac{1}{6}c_1 \quad c_5 = \frac{-3}{20}c_3 = -\frac{1}{40}c_0 \\ c_7 = -\frac{7}{42}c_5 = \frac{1}{240}c_1 \quad \ldots \end{cases}$$

$$\Rightarrow y(x) = c_0 \times \left(1 + \frac{3}{2}x^2 - \frac{1}{8}x^4 + \frac{1}{48}x^6 + \ldots\right)$$
$$+ c_1 \times \left(x + \frac{1}{6}x^3 - \frac{1}{40}x^5 + \frac{1}{240}x^7 + \ldots\right)$$

(2) $y''(x) + 2y'(x) - y(x) = 0$

ASSUME: $y(x) = \sum_{n=0}^{\infty} c_n x^n$

Start each sum's index @ 1st non-zero term in sum.

$$\hookrightarrow 0 = \sum_{n=0}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=0}^{\infty} 2n c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n$$

$$= \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=1}^{\infty} 2n c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n$$

n=0,1 terms are zero

n=0 term is zero

Re-index 1st & 2nd sum

$$= \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n + \sum_{n=0}^{\infty} 2(n+1) c_{n+1} x^n - \sum_{n=0}^{\infty} c_n x^n$$

$$= \sum_{n=0}^{\infty} \left[(n+2)(n+1) c_{n+2} + 2(n+1) c_{n+1} - c_n \right] x^n$$

$$\hookrightarrow (n+2)(n+1) c_{n+2} + 2(n+1) c_{n+1} - c_n = 0$$

$n=0$: $2c_2 + 2c_1 - c_0 = 0 \rightarrow c_2 = \frac{1}{2} c_0 - c_1$

$n=1$: $6c_3 + 4c_2 - c_1 = 0 \rightarrow c_3 = \frac{1}{6} c_1 - \frac{2}{3} c_2 = \frac{1}{6} c_1 - \frac{1}{3} c_0 + \frac{2}{3} c_1$

$$= -\frac{1}{3} c_0 + \frac{5}{6} c_1$$

$n=2$: $12 c_4 + 6 c_3 - c_2 = 0 \rightarrow c_4 = \frac{1}{12} c_2 - \frac{1}{2} c_3 = \frac{1}{24} c_0 - \frac{1}{12} c_1 + \frac{1}{6} c_0 - \frac{5}{12} c_1$

$$= \frac{5}{24} c_0 - \frac{1}{2} c_1$$

$$\Rightarrow y(x) = c_0 \times \left(1 + \frac{1}{2}x^2 - \frac{1}{3}x^3 + \frac{5}{24}x^4 + \dots\right) + c_1 \times \left(x - x^2 + \frac{5}{6}x^3 - \frac{1}{2}x^4 + \dots\right)$$

(3) $y''(x) + y'(x) - x^2 y(x) = 0$

$$y(x) = \sum_{n=0}^{\infty} c_n x^n$$

$$\hookrightarrow 0 = \underbrace{\sum_{n=0}^{\infty}}_{n=0,1 \text{ terms vanish}} n(n-1) c_n x^{n-2} + \underbrace{\sum_{n=0}^{\infty}}_{n=0 \text{ term vanishes}} n c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^{n+2}$$

$$= \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=1}^{\infty} n c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^{n+2}$$

The 1st two sums have powers $x^0, x^1, x^2, \ldots$, while the last sum starts @ x^2. Let's write out the x^0 & x^1 terms explicitly:

$$0 = 2 \cdot c_2 x^0 + 6 c_3 x^1 + \sum_{n=4}^{\infty} n(n-1) c_n x^{n-2} + c_1 x^0 + 2 c_2 x^1 + \sum_{n=3}^{\infty} n c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^{n+2}$$

$$= (2c_2 + c_1) x^0 + (6c_3 + 2c_2) x^1 + \underbrace{\sum_{n=4}^{\infty} n(n-1) c_n x^{n-2}}_{n \to n+4} + \underbrace{\sum_{n=3}^{\infty} n c_n x^{n-1}}_{n \to n+3} - \sum_{n=0}^{\infty} c_n x^{n+2}$$

$$\sum_{\underline{n+4=4}}^{\infty} (n+4)(n+3) c_{n+4} x^{n+2} \qquad \sum_{\underline{n+3=3}}^{\infty} (n+3) c_{n+3} x^{n+2}$$

$n=0$ $\qquad$ $n=0$

$$= (2c_2 + c_1) x^0 + (6c_3 + 2c_2) x^1 + \sum_{n=0}^{\infty} \left[(n+4)(n+3) c_{n+4} + (n+3) c_{n+3} - c_n \right] x^{n+2}$$

So we have: $\qquad$ True for all $n = 0, 1, 2, \ldots$

$$c_2 = -\tfrac{1}{2} c_1 \qquad c_3 = -\tfrac{1}{3} c_1 = \tfrac{1}{6} c_1 \qquad (n+4)(n+3) c_{n+4} + (n+3) c_{n+3} - c_n = 0$$

$n=0$: $\quad 12 c_4 + 3 c_3 - c_0 = 0 \rightarrow c_4 = \frac{1}{12} c_0 - \frac{1}{4} c_3 = \frac{1}{12} c_0 - \frac{1}{24} c_1$

$n=1$: $\quad 20 c_5 + 4 c_4 - c_1 = 0 \rightarrow c_5 = \frac{1}{20} c_1 - \frac{1}{5} c_4 = \frac{1}{20} c_1 - \frac{1}{60} c_0 + \frac{1}{120} c_1$

$$= -\frac{1}{60} c_0 + \frac{7}{120} c_1$$

$n=2$: $\quad 30 c_6 + 5 c_5 - c_2 = 0 \rightarrow c_6 = \frac{1}{30} c_2 - \frac{1}{6} c_5 = -\frac{1}{60} c_1 + \frac{1}{360} c_0 - \frac{7}{720} c_1$

$$= \frac{1}{360} c_0 - \frac{19}{720} c_1$$

$$\Rightarrow y(x) = c_0 \times \left(1 + \frac{1}{12} x^4 - \frac{1}{60} x^5 + \frac{1}{360} x^6 + \ldots \right)$$
$$+ c_1 \times \left(x - \frac{1}{2} x^2 + \frac{1}{6} x^3 - \frac{1}{24} x^4 + \frac{7}{120} x^5 - \frac{19}{720} x^6 + \ldots \right)$$